Flow with Nonlinear Potential in General Networks

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Abstract

The aim of this paper is a short survey of models and methods using to optimize general networks with nonlinear non-convex restrictions and objectives possessing mixed continuous-discrete optimization variables. There are discussed the problem formulations and solution methods for simulation, optimization, sensitivity and stability analysis for flow with nonlinear potential in general networks. These problems and the developed methods have industrial application e.g. by gas networks.

classifications:] networks, flow, nonlinear potential, nonlinear programming. integer programming, simulation, optimization.

1 Introduction

There are 2 approaches to describe flow in real and virtual networks presently. The first approach is based on a physical description of flow laws. Conservation laws of mass, impulse and energy resulting in Ohm law for electrical current or Bernoulli law for fluid flow, and thermic state equation are used there. This approach led to physical and technical modelling and simulation of real existing systems. Dynamic optimization problem is not solved. Instead of dynamic optimization, 'try and error' method based on simulation was proposed.

The second approach is developed more for economics. It is based on graph theory and special methods of linear programming. The flow conservation law represents a main part of restrictions in a mathematical model there. Such mathematical problems as maximal flow in network and minimal cost network flow could be mentioned there. In their dual problems, linear potential is appeared. Linear potential is connected with flow by means of linear analogue of Ohm law for passive electric chains.

However, there exist technical systems within which the connection between flow and potential is not linear. The objective function there depends not only either on flow or on potential, but on both of them. There are non-convex and non-smooth functions among constraints and objectives. The pipe networks, energy systems and electrical chains with nonlinear associated facilities are examples for such large - scale technical systems. There are some working approaches to optimize such systems [5], [11], [8] but the optimization theory based on dynamic models is not developed for general networks with nonlinear connection between flow and potential.

The aim of this paper is a short survey of models and methods using to optimize general networks with nonlinear non-convex restrictions and objectives possessing mixed continuous-discrete optimization variables. There are discussed the problem formulations and solution methods for simulation, large-scale nonlinear discrete-continuous network optimization, sensitivity and stability analysis and reliability study for flow with nonlinear potential in general networks.

The mentioned above problems find an industrial application. The developed methods are realized in software called ACCORD which is in operation in different countries.

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2 Mathematical model and problem formulation

2.1 Passive and active elements

A passive element is a two-pole that does not need an additional energy supply to work. As simple examples, the resistors and pipe could be mention. A dynamic model of a pipe could be written as a system of algebraic and partial differential equations (PDE) for state variable vector $(\rho, T, p, v, h)^T$:

(1) $$\rho = \rho(p,h) := 1ptpzRT, wherez = z(p,T)$$

(2) $$h = c_p T, wherec_p = c_p(p,T)$$

(3) $$1pt\partial\partial t \begin{pmatrix} \rho \\ \rho v \\ \rho\left(h + 1ptv^2 2\right) - p \end{pmatrix} + 1pt\partial\partial x \begin{pmatrix} \rho v \\ \rho v^2 + p \\ \rho v\left(h + 1ptv^2 2\right) \end{pmatrix} = \begin{pmatrix} 0 \\ -1pt\lambda D\rho 1ptv\,|v|2 - \rho g 1ptdHdx \\ 1pt\Omega S - v\rho g 1ptdHdx \end{pmatrix}$$

where c_p - specific heat by constant pressure $[J/kg\ K]$; $\dot{M}$ - mass flowrate $[kg/s]$; g - gravitational acceleration $\left[m/s^2\right]$; h - specific ehthalpy $[J/kg]$; F - objective function; H - altitude $[m]$; p - pressure $\left[N/m^2\right]$; R - individual gas constant $[J/kg\ K]$; ρ - specific mass (density) $\left[kg/m^3\right]$; S - cross section $\left[m^2\right]$; T - temperature $[K]$; t - time coordinate $[s]$; v - flow velocity $[m/s]$; x - length coordinate $[m]$; z - compressibility factor $[-]$; Ω - heat flow per length $[W/m]$. The equations express respectively thermic and calorific state equations and conservation laws of mass, impulse, and energy. The PDE is of hyperbolic type. For a practical simulation, a mixing from initial and boundary conditions is used.

For a lot of considerations, a steady state model of a passive element is precise enough. The model is algebraic and could be obtain after simplification from (3) as follows:

(4) $$q(x) = Const, \quad i \le x \le k$$

(5) $$f(p_i, p_k, q) = 0, \quad where \quad p_i = p\,|_{x=i}, \quad p_k = p\,|_{x=k}$$

As example of active element, an amplifier, a pump, and a compressor could be mentioned. Their common feature is that they have a plane simply connected operational range [9]. This range must be taken as restriction in model. Unfortunately, the range forms a non-convex domain mostly.

In some cases, in particular for hydraulic networks, active elements react much quicker than passive elements. Then steady state models could be used for active elements by network modelling. These steady state models are algebraic.

Together with flow balance equations, the equations and inequalities describing models of active and passive elements form a part of restrictions by network modelling.

2.2 Steady state continuous-discrete optimization problem on general network

Let be given a network $G = (V, E)$ with a set of nodes V and a set of edges E. Let q_{ik} is a flow in the edge (i,k) and Q_i is supply or demand called intensity of node i. It means that

(6) $$\sum_{k \in V} q_{ik} + Q_i = 0, \quad i \in V,$$

(7) $$q_{ik} = -q_{ki}, \quad (i,k) \in E.$$

Suppose that there are given families of functional dependencies on flow and potentials:

(8) $$f_{d_{ik}}(p_i, p_k, q_{ik}, c_{ik}) = 0, \quad (i,k) \in E;$$

(9) $$d_{ik} \in \{1, ..., N_{ik}\}, \quad (i,k) \in E.$$

Here c_{ik} is a vector of continuous parameters (coefficients), and d_{ik} is a discrete parameter on the edge (i,k). Suppose that there are given limitations Q_i^- , Q_i^+, p_i^- , p_i^+, c_{ik}^-, c_{ik}^+ :

(10) $$Q_i^- \le Q_i \le Q_i^+, \quad i \in V;$$

(11) $$p_i^- \le p_i \le p_i^+, \quad i \in V;$$

(12) $$c_{ik}^- \le c_{ik} \le c_{ik}^+, \quad (i,k) \in E;$$

and the other restrictions can be represented by inequalities with given a^-_{ik}, a^+_{ik} for the functions $a_{ik}(p_i, p_k, q_{ik}, c_{ik}, d_{ik})$ which have to be calculated:

$$a^-_{ik} \leq a_{ik}(p_i, p_k, q_{ik}, c_{ik}, d_{ik}) \leq a^+_{ik}, \quad (i,k) \in E. \tag{13}$$

The considering objective function depends both on flow q_{ik} and on potentials p_i, p_k, on intensities Q_i, continuous c_{ik} and discrete parameters d_{ik}. The problem consists in

$$minimize F = \sum_{(i,k)\in E} F^{(1)}_{ik}(p_i, p_k, q_{ik}, c_{ik}, d_{ik}) + \sum_{i\in V} F^{(2)}_i(p_i, Q_i) \tag{14}$$

$$subjectto (6 - 13). \tag{15}$$

The set of available values of discrete parameters d_{ik} in (9) means that a family of functions $f_{d_{ik}}$ can act on the edge (i, k), and we have to select the best form of every equation (8). Thus, we have to select the best equation (8) from the point of view of functional (14). We may interpret this as a selection of the most profitable equipment which is installed or can be installed on the place (i, k).

The continuous parameters c_{ik} in (8) can be interpreted as parameters that smoothly regulate the work of equipment d_{ik} in bounds (12). We may interpret (13) as restrictions on the power, temperature, dissipation and other characteristics of the equipment that is represented on the edge (i, k) . The inequalities (10) and the dependence of objective function on intensities Q_i mean that the most profitable values of supplies and demands have to be chosen.

The functions $F^{(1)}_{ik}, f_{d_{ik}}, a_{ik}$ staying in the objective function and in the restrictions are non-linear and can be non-smooth and non-convex but expensive to compute. Usually the number of nodes and edges are hundreds with a tendency to be thousands. So the formulated problem is a problem of large-scale nonlinear discrete-continuous programming on general networks. This problem generalize the well-known minimal cost network flow problem.

3 Methods of the discrete-continuous nonlinear network optimization

The formulated problem (6 - 14) is a problem for searching a conditional extremum for a function with continuous-discrete parameters of optimization. According to the mixing character of variables, the combination of the continuous and discrete optimization methods is used for the problem solving. The integer and nonlinear programming and graph theory are basis of the offered method. Its main characteristic feature consists in the obtaining of dominant solutions on fragments of the network.

3.1 Simple examples

The next 2 examples and orders are dual each other. Let discrete parameters are fixed.

Let a node v_{root} is chosen. There exists a linear order ℓ_p on V such that if we enumerate the nodes with respect to this linear order then: 1) the root has the number $n_{v_{root}} = 1$; 2) for every node with number $n > 1$ there is just one adjacent node with number $n_u < n$. Other words, it is possible to construct an optimization process so that among potentials p_i only p_{root} is considered as an independent (called non-basic) variable , and all other p_i are dependent (called basic) variables computing after p_{root} and q_{ik}.

There exists a linear order ℓ_q on nodes V such that if we enumerate the nodes with respect to this linear order then for every node with number n the numbers of all adjacent nodes except may be one are less then $n : n_u < n$. Other words, it is possible to construct an optimization process so that among flows q_{ik} only non-tree flows are considered as independent variables (or non-basic variables in terminology of linear programming), and all other flows q_{ik} are dependent (or basic) variables.

3.2 The continuous dominant problems

Let D and X are correspondingly the sets of discrete and continuous independent variables. Denote a problem for searching a feasible solution (6 - 13) by $P(D, X)$ and an optimization problem (6 - 14) by $FP(D, X)$.

Problem $P\hat{}$ dominates another one P and is called as a dominant for P, if the feasible set for $P\hat{}$ contains such set for P.

Lemma 3.1 *For any discrete-continuous problem $P(D, X)$ there exists a continuous dominant $P\hat{}(X_D, X)$, where a set of discrete variables D is replaced by a set of continuous variables X_D.*

Proof On the edges with a discrete variable d_{ik}, the equation $t_{ik} = p_i - p_k$ replaces the family of equations (8 - 9). A component t_{ik} of tension vector replaces the discrete variable d_{ik} and the continuous ones c_{ik} there. ■

As consequence, there is the continuous dominant $FP\hat{}(X_D, X)$ for an initial optimization problem $FP(D, X)$.

Assign a node where potential is either given or varied as a root. Let a set of discrete variables D is participated onto two classes D' and D^C. Collections $d' \in D'$ are called as fragments of $d \in D$. Consider an initial discrete-continuous problem (6 - 13) by fixed $d' \in D'$; denote it $P_{d'}(D^C, X)$. We can construct for a problem $P_{d'}(D^C, X)$ its continuous dominant $P_{d'}\hat{}(X_{D^C}, X)$. The problem $P_{d'}\hat{}$ defines a feasibility of fragment d'. Fragment d' is infeasible, if $P_{d'}\hat{}$ has no solution. The explicit definition of fragment d' is based on the follows

Lemma 3.2 *Let $G = (V, E)$ is a connected graph, $v_0 \in V$ is a node, and a finite set $M_0 \subset E$ consists of $|M_0|$ edges. Consider M as a set of subsets of M_0 and write down $d \in M$ as*

(16) $$d = (d_1, ..., d_{|M_0|})_\rho \quad , \qquad d_i = 0, 1.$$

for a linear order ρ on M_0. There exists such a linear order π on M_0 that for every m-length fragment $d^m = (d_1, ..., d_m, 0, ..., 0)_\pi$ there is a connected subgraph $G_i = (V_i, E_i), i = i(m)$, that is a minimal connected subgraph containing the node v_0, all the edges from d^m, and all their endnodes.

Proof A proof is constructive. It consists of the following steps:
1) Construction a linear order ℓ_v on a rooted spanning tree;
2) Construction a linear order π on the subset M_0 of edges;
3) Construction a minimal subgraph for a set d^m of the first m edges from M_0 . ■

Let the discrete variables are acting only on edges. Take the set of edges possessing discrete variables in the role of M_0. Let an edge $e \in M_0$ has a number j in a linear order π; denote it as e_j. With the notation (16) for $d \subseteq M_0$ and $d^m = (d_1, ..., d_m, 0, ..., 0)$ for m - length fragment, let us allow to use a non-null value of a discrete variable acting on the edge $e_j \in d$ as d_j, and to use $d_j = 0$ for $e_j \notin d$. In the last case we shall say that discrete variable is not given there. So the linear order π on M_0 induces a linear order on a set D of discrete variables, and we can tell about m-length fragment of discrete variables. The lemma 3.2 can be reformulated as follows.

Theorem 3.1 *Let $G = (V, E, v_{root})$ is connected network with one node marked as v_{root}. Let some edges of the network possess discrete variables, and the last ones are only there. There exists a linear order on the set of discrete variables D that:*

1) every $d \in D$ can be written as

(17) $$d = (d_1, ..., d_M), \quad d_i = 0, 1, ..., N_i,$$

where $d_i = 0$ means that value of corresponding variable is not given;

2) for every m-length fragment $d^m = (d_1, ..., d_m, 0, ..., 0)$ there exists a connected sub-network $G_i = (V_i, E_i), i = i(m)$, that is a minimal sub-network containing the root, all the edges with non-null discrete variables from d^m, and all endnodes of these edges.

The solving method for the continuous-discrete problem $FP(D, X)$ can be presented as a branching multi-level computational process. On the main branch of this process we solve the continuous problem $FP\hat{}(D, X)$ or $P(D, X)$.

3.3 A method for solving the continuous problems on general networks

In continuous problems $FP(X)$, $FP\hat{}(X_D, X)$, and $P_{d'}\hat{}$, the components of continuous variable vector $x = (x_1, ..., x_n)$ are divided into 3 groups: those from the initial problem $FP(D, X)$; the flows on chords; and the continuous variables which change the discrete ones in the dominant continuous search. The flows on chords are of a special meaning. They can be computed as a solution of the equation system (6),(8),(9) by the fixed both discrete and continuous variables of 2 other groups. Then the dimension of continuous variables and of equation systems (6),(8) can be reduced thanks to the direct solving of such systems on a tree.

To solve the continuous search problem, the classical methods of constrained and unconstrained optimization from nonlinear programming are implemented. The preference is given to the methods without derivatives because of non-convex and non-smooth functions in restrictions and objectives.

Components of objective, penalty and Lagrangian functions are computed as soon as a network state component s_i is been known, $s = (q_{ik}, Q_i, p_i, c_{ik}, d_{ik})$.

By computation of state s, we have to distribute flow and potential. To effectively solve the equation systems (6) and (8) on tree, respectively the width- and depth-first searches on graph are used.

3.4 Modification of the branch-and-bound method

In our modification of the branch-and-bound method, only m-length fragments are used. If fragment is infeasible then all its extension $(d_1, ..., d_m, j_{m+1}, ..., j_M)$ are excluded from the further examination. The examination is gone in a lexicographic order of vectors $d = (d_1, ..., d_M)$. Starting from $d^{(0)} = (0, ..., 0)$, a transition from $d^{(n)}$ to $d^{(n+1)}$ is performed by the next rule: if $d^{(n)} = d^m$ is fragment with length $m < M$ and $d^{(n)}$ is feasible then

$$d^{(n+1)} = (d_1, ..., d_m, 1, 0, ..., 0) \tag{18}$$

else $d^{(n+1)} = d^m \bigoplus e^m$. Here $e^m = (0, ..., 1, ...0)$ and

$$d^m \bigoplus e^m = \begin{cases} (d_1, ..., d_{m-1}, d_m + 1, 0, ..., 0), & if d_m < N_m; \\ (d_1, ..., d_{\mu-1} + 1, 0, ..., 0), & if d_m = N_m and \exists \mu : 1 < \mu \le m : \\ & d_{\mu-1} < N_{\mu-1} \& d_k = N_k, k = \mu \ldots m; \\ (0, ..., 0), & if d_k = N_k, \forall k = 1, \ldots, M. \end{cases} \tag{19}$$

3.5 An other procedure for searching an optimal integer-feasible solution

The idea of an other procedure for $FP(D, X)$ is connected with a possibility to check, if there is a realization of equation $t_{ik} = p_i - p_k$ in the equation family (8),(9). This sub-problem is to construct mapping

$$\{(t_{ik}, p_i, p_k, q_{ik}) \mid t_{ik} = p_i - p_k\} \longrightarrow \{(d_{ik}, c_{ik}, p_i, p_k, , q_{ik}) \mid (8), (9), (12)\}. \tag{20}$$

The second procedure for solving $FP(D, X)$ is proposed as follows. The dominant problem $FP\hat{}(X_D, X)$ is to be solved. On the every step of continuous optimization $FP\hat{}(X_D, X)$, there is checked the existence of feasible discrete variable d_{ik} and perhaps continuous ones c_{ik} for every continuous variable t_{ik} changing d_{ik} in the dominant problem. If there is no such feasible discrete variable d_{ik} then the corresponding continuous variable t_{ik} would be penalized. Other words, t_{ik} is penalized if there is no mapping (20). Then an optimum of problem $FP\hat{}(X_D, X)$ brings an optimum for the initial problem $FP(D, X)$.

The difference between this method and the branch-and-bound approach is clear for the cases of pure discrete problems on a tree. Such class of problems means that graph $G = (V, E)$ is a tree and there is no continuous variables at all: $X = \emptyset$. Then the formulated version of branch-and-bound method can be very efficient. It brings a global optimum as a solution.

For the same problem $FP(D)$ on a tree, the second method based on the checking of integer-feasibility by the solving of dominant problem $FP\hat{}(X_D)$ can fall in a local optimum. To avoid it, a restart can be used. By restart, it is worth to move the root in an other node, if the restrictions (11) allow it.

However, for the network with complicated topology the branch-and-bound method could be expensive if not prohibited. This is valid especially for the networks with a lot of cycles. Then the second procedure with penalizing non-realizable t_{ik} is available there. In practice, the method avoids the local optima.

4 Sensitivity and stability analysis for reliability study

There are 2 extreme approaches by analysis either a network state is stable. The first one is used by operating when control vector is chosen (fixed) .and it is necessary to check either this control vector still lead to a feasible network state if initial or boundary conditions are changing. The second one is used by design when it is necessary to check either there is a value of control vector for every possible operating point. The first approach we shall call the strong stability while the second one we shall call the weak stability.

Let be given an interval of analyzing parameter $\pi_0 - \delta \le \pi \le \pi_0 + \delta$. The network state x is called strong stable with respect to a parameter π by fixed control u_0 if there is an interval $(\pi_0 - \Delta, \pi_0 + \Delta) \supseteq (\pi_0 - \delta, \pi_0 + \delta)$ that for every $\pi \in (\pi_0 - \Delta, \pi_0 + \Delta)$ the network state $x(u_0, \pi)$ is feasible.

Let be given an interval of analyzing parameter $\pi_0 - \delta \le \pi \le \pi_0 + \delta$, a neighborhood U_0 of a control u_0 and $\eta > 0$, and let $F(u) = F(u, x(u, \pi))$ is objective function. The network state x is called weak stable with respect to a parameter π if there is an interval $(\pi_0 - \Delta, \pi_0 + \Delta) \supseteq (\pi_0 - \delta, \pi_0 + \delta)$ that for every $\pi \in (\pi_0 - \Delta, \pi_0 + \Delta)$ there is a control $u \in U_0$ which provides that the network state $x(u, \pi)$ is feasible and that

$$\left| F(u_0, x(u_0, \pi) - \min_{u \in U_0} F_1(u, x(u, \pi)) \right| < \eta \tag{21}$$

where $F_1(u,x) = F(u,x) + \psi(u)$ and $\psi(u)$ is a cost of control switch.

For a scalar parameter π, it is possible to compute the above Δ showing a feasibility interval with respect to parameter π. For a set of analyzing parameters $\pi = (\pi_1, ..., \pi_n)$, a method of statistic tests could be used. If for the most points $\pi \in \{\pi_{i0} - \delta_i \le \pi_i \le \pi_{i0} + \delta_i\}$ (saying, for 95 %) there is a control $u \in U_0$ providing feasibility of the network state $x(u,\pi)$ and condition (21) then network state is called weak stable with respect to parameter set π. The definition of strong stability with respect to parameter set π is evident.

In complement to statistic approach, one simple result for 'definite' approach has to be mentioned. The capacity q^+_{ik} of an edge (i,k) can be found from equation (8) varying p_i, p_k, d_{ik} :

$$q^+_{ik} := \max_{d_{ik},\ p_i, p_k} \{q_{ik} \mid f_{d_{ik}}(p_i, p_k, q_{ik}) = 0, \quad d_{ik} \in \{1, ..., N_{ik}\}, \quad p_n \in [p^-_n, p^+_n], \quad n = i, k\} \tag{22}$$

In some cases we can precise potential intervals $[p^-_i, p^+_i]$ in (11) and capacities therefore.

Lemma 4.1 *Let be given intervals* $[p^-_i, p^+_i]$ *(11), functions* $f_{d_{ik}}$ *in models (8) are monotonous relative both potentials* p_i, p_k *and* q_{ik} *separately, and every model* $f_{d_{ik}}$ *in (8) is chosen:* $d_{ik} = Const, \quad (i,k) \in E$, *i.e.* $f_{d_{ik}} = f_{ik}$ *For a known flow* q_{ik} *, it can be found such intervals*

$$[p^{\min}_i, p^{\max}_i] \subseteq [p^-_i, p^+_i] \tag{23}$$

that the inequality system (8),(11) has a solution if and only if the system (8),(23) has a solution.

5 Conclusions

The solution methods and their implementation are described for a class of problems of large-scale nonlinear discrete-continuous network optimization on general networks. Objective function depends both on flow and potential. Minimal cost and maximal flow problems are generalized. The problem consists both in the optimal choice of dependencies on flow and potential from the families given on some edges and in optimization of values of intensities, flow, and potential.

The formulations, methods and algorithms are developed for the following problems: 1) the (pure) continuous nonlinear optimization on general network with a given functional dependence between flow and potentials; 2) the (pure) discrete nonlinear optimization problem of selection the best functional dependence between flow and potentials from given families; 3) the (mixed) continuous-discrete nonlinear optimization on general network with given families of dependences between flow and potentials. The developed optimization method represents a branching multi-level computational process. It is based on nonlinear and integer programming and on graph theory. Its main characteristic feature consists in obtaining of dominant solutions on network fragments. During optimization, the offered methods adapt themselves to the structure of every network and to the features of both objective function and variables. It allows to use such objectives as minimization of cost, of expenditure, and of set-point deviation, maximization of flow, of profit, and so on.

On the base of the developed optimization methods, some approaches for sensitivity analysis, stability investigation and reliability study are proposed.

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